

TIME-HOMOGENEOUS PARABOLIC WICK-ANDERSON MODEL IN ONE SPACE DIMENSION: REGULARITY OF SOLUTION

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ABSTRACT. Even though the heat equation with random potential is a well-studied object, the particular case of time-independent Gaussian white noise in one space dimension has yet to receive the attention it deserves. The paper investigates the stochastic heat equation with space-only Gaussian white noise on a bounded interval. The main result is that the space-time regularity of the solution is the same for additive noise and for multiplicative noise in the Wick-Itô-Skorokhod interpretation.

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1. INTRODUCTION

Consider the stochastic heat equation

$$\frac{\partial u(t, x)}{\partial t} = \frac{\partial^2 u(t, x)}{\partial x^2} + u(t, x)\dot{W}, \quad (1.1)$$

where \dot{W} is a Gaussian white noise. Motivated by various applications in physics, equation (1.1) is often called parabolic Anderson model with continuous time and space parameters.

If $W = W(t)$ is a Brownian motion in time, then, with an Itô interpretation, a change of variables $u(t, x) = v(t, x) \exp(W(t) - (t/2))$ reduces (1.1) to the usual heat equation $v_t = v_{xx}$.

If $W = W(t, x)$ is a two-parameter Brownian motion, or Brownian sheet, then equation (1.1) has been studied in detail, from one of the original references [18, Chapter 3] to a more recent book [8]. In particular, the Itô interpretation is the only option; cf. [3].

If $W = W(x)$ is a Brownian motion in space, then equation (1.1) has two different interpretations:

(1) Wick-Itô-Skorokhod interpretation

$$\frac{\partial u(t, x)}{\partial t} = \frac{\partial^2 u(t, x)}{\partial x^2} + u(t, x) \diamond \dot{W}(x), \quad (1.2)$$

where \diamond is the Wick product;

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(2) Stratonovich interpretation

$$\frac{\partial u(t, x)}{\partial t} = \frac{\partial^2 u(t, x)}{\partial x^2} + u(t, x) \cdot \dot{W}(x), \quad (1.3)$$

where $u(t, x) \cdot \dot{W}(x)$ is understood in the point-wise, or path-wise, sense.

In [5, 6], equation (1.2) is studied on the whole line as a part of a more general class of equations. Two works dealing specifically with (1.2) are [17], where the equation is considered on the whole line, and [15], where the Dirichlet boundary value problem is considered with a slightly more general random potential.

According to [17, Theorem 4.1], the solution of (1.2) is almost Hölder(1/2) in time and space. By comparison, the solution of (1.3) is almost Hölder(3/4) in time and almost Hölder(1/2) in space [6, Theorem 4.12], whereas for the equation with additive noise

$$\frac{\partial u(t, x)}{\partial t} = \frac{\partial^2 u(t, x)}{\partial x^2} + \dot{W}(x), \quad t > 0, \quad x \in \mathbb{R}, \quad u(0, x) = 0,$$

the solution is almost Hölder(3/4) in time and is almost Hölder(3/2) in space, which follows by applying the Kolmogorov continuity criterion to

$$u(t, x) = \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi s}} e^{-(x-y)^2/(4s)} dW(y) ds.$$

The objective of this paper is to establish optimal space-time regularity of the solution of

$$\begin{aligned} \frac{\partial u(t, x)}{\partial t} &= \frac{\partial^2 u(t, x)}{\partial x^2} + u(t, x) \diamond \dot{W}(x), \quad t > 0, \quad 0 < x < \pi, \\ u_x(t, 0) &= u_x(t, \pi) = 0, \quad u(0, x) = u_0(x), \end{aligned} \quad (1.4)$$

and to define and investigate the corresponding fundamental solution. We show that the solution of (1.4) is almost Hölder(3/4) in time and is almost Hölder(3/2) in space. As a result, similar to the case of space-time white noise, solutions of equations driven by either additive or multiplicative Gaussian white noise in space have the same regularity, justifying the optimality claim in connection with (1.4).

Our analysis relies on the chaos expansion of the solution and the Kolmogorov continuity criterion. Section 2 provides the necessary background about chaos expansion and the Wick product. Section 3 introduces the chaos solution of (1.4). Section 4 establishes basic regularity of the chaos solution as a random variable and introduces the main tools necessary for the proof of the main result. Section 5 establishes the benchmark regularity result for the additive-noise version of (1.4). The main results, namely, Hölder continuity of the chaos solution of (1.4) in time and space, are in Sections 6 and 7, respectively. Section 8 is about the fundamental chaos solution of (1.4). Section 9 discusses various generalizations of (1.4), including other types of boundary conditions.

We use the following notations:

$$f_t(t, x) = \frac{\partial f(t, x)}{\partial t}, \quad f_x(t, x) = \frac{\partial f(t, x)}{\partial x}, \quad f_{xx}(t, x) = \frac{\partial^2 f(t, x)}{\partial x^2};$$

$$\mathbb{T}_{s,t}^n = \{(s_1, \dots, s_n) \in \mathbb{R}^n : s < s_1 < s_2 < \dots < s_n < t\},$$

$$0 \leq s < t, \quad n = 1, 2, \dots;$$

$$(g, h)_0 = \int_0^\pi g(x)h(x)dx, \quad \|g\|_0 = \sqrt{(g, g)_0}, \quad g_k = (g, \mathbf{m}_k)_0,$$

where $\{\mathbf{m}_k, k \geq 1\}$ is an orthonormal basis in $L_2((0, \pi))$;

$$dx^n = dx_1 dx_2 \cdots dx_n.$$

2. THE CHAOS SPACES

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A **Gaussian white noise** \dot{W} on $L_2((0, \pi))$ is a collection of Gaussian random variables $\dot{W}(h)$, $h \in L_2((0, \pi))$, such that

$$\mathbb{E}\dot{W}(g) = 0, \quad \mathbb{E}(\dot{W}(g)\dot{W}(h)) = (g, h)_0. \quad (2.1)$$

For a Banach space X , denote by $L_p(W; X)$, $1 \leq p < \infty$, the collection of random elements η that are measurable with respect to the sigma-algebra generated by $\dot{W}(h)$, $h \in L_2((0, \pi))$, and such that $\mathbb{E}\|\eta\|_X^p < \infty$.

In what follows, we fix the Fourier cosine basis $\{\mathbf{m}_k, k \geq 1\}$ in $L_2((0, \pi))$:

$$\mathbf{m}_1(x) = \frac{1}{\sqrt{\pi}}, \quad \mathbf{m}_k(x) = \sqrt{\frac{2}{\pi}} \cos(kx), \quad (2.2)$$

and define

$$\xi_k = \dot{W}(\mathbf{m}_k). \quad (2.3)$$

By (2.1), ξ_k , $k \geq 1$, are iid standard Gaussian random variables, and

$$\dot{W}(h) = \sum_{k \geq 1} (\mathbf{m}_k, h)_0 \xi_k.$$

As a result,

$$\dot{W}(x) = \sum_{k \geq 1} \mathbf{m}_k(x) \xi_k \quad (2.4)$$

becomes an alternative notation for \dot{W} ; of course, the series in (2.4) diverges in the traditional sense.

It follows from (2.1) that $W(x) = \dot{W}(\chi_{[0,x]})$ is a standard Brownian motion on $[0, \pi]$, where $\chi_{[0,x]}$ is the indicator function of the interval $[0, x]$.

Denote by \mathcal{J} the collection of multi-indices $\boldsymbol{\alpha}$ with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots)$ so that each α_k is a non-negative integer and $|\boldsymbol{\alpha}| := \sum_{k \geq 1} \alpha_k < \infty$. For $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{J}$, we define

$$\boldsymbol{\alpha} + \boldsymbol{\beta} = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots), \quad \boldsymbol{\alpha}! = \prod_{k \geq 1} \alpha_k!.$$

Also,

- $(\mathbf{0})$ is the multi-index with all zeroes;
- $\boldsymbol{\epsilon}(i)$ is the multi-index $\boldsymbol{\alpha}$ with $\alpha_i = 1$ and $\alpha_j = 0$ for $j \neq i$;
- $\boldsymbol{\alpha} - \boldsymbol{\beta} = (\max(\alpha_1 - \beta_1, 0), \max(\alpha_2 - \beta_2, 0), \dots)$;

$$\bullet \alpha^-(i) = \alpha - \epsilon(i).$$

An alternative way to describe a multi-index $\alpha \in \mathcal{J}$ with $|\alpha| = n > 0$ is by its **characteristic set** K_α , that is, an ordered n -tuple $K_\alpha = \{k_1, \dots, k_n\}$, where $k_1 \leq k_2 \leq \dots \leq k_n$ indicate the locations and the values of the non-zero elements of α : k_1 is the index of the first non-zero element of α , followed by $\max(0, \alpha_{k_1} - 1)$ of entries with the same value. The next entry after that is the index of the second non-zero element of α , followed by $\max(0, \alpha_{k_2} - 1)$ of entries with the same value, and so on. For example, if $n = 7$ and $\alpha = (1, 0, 2, 0, 0, 1, 0, 3, 0, \dots)$, then the non-zero elements of α are $\alpha_1 = 1$, $\alpha_3 = 2$, $\alpha_6 = 1$, $\alpha_8 = 3$, so that $K_\alpha = \{1, 3, 3, 6, 8, 8, 8\}$: $k_1 = 1$, $k_2 = k_3 = 3$, $k_4 = 6$, $k_5 = k_6 = k_7 = 8$.

Define the collection of random variables $\Xi = \{\xi_\alpha, \alpha \in \mathcal{J}\}$ by

$$\xi_\alpha = \prod_k \left(\frac{H_{\alpha_k}(\xi_k)}{\sqrt{\alpha_k!}} \right),$$

where ξ_k is from (2.3) and

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \quad (2.5)$$

is the Hermite polynomial of order n . By a theorem of Cameron and Martin [1], Ξ is an orthonormal basis in $L_2(W; X)$ as long as X is a Hilbert space. Accordingly, in what follows, we always assume that X is a Hilbert space.

For $\eta \in L_2(W; X)$, define $\eta_\alpha = \mathbb{E}(\eta \xi_\alpha) \in X$. Then

$$\eta = \sum_{\alpha \in \mathcal{J}} \eta_\alpha \xi_\alpha, \quad \mathbb{E} \|\eta\|_X^2 = \sum_{\alpha \in \mathcal{J}} \|\eta_\alpha\|_X^2.$$

We will often need spaces other than $L_2(W; X)$:

- The space

$$\mathbb{D}_2^n(W; X) = \left\{ \eta = \sum_{\alpha \in \mathcal{J}} \eta_\alpha \xi_\alpha \in L_2(W; X) : \sum_{\alpha \in \mathcal{J}} |\alpha|^n \|\eta_\alpha\|_X^2 < \infty \right\}, \quad n > 0;$$

- The space

$$L_{2,q}(W; X) = \left\{ \eta = \sum_{\alpha \in \mathcal{J}} \eta_\alpha \xi_\alpha \in L_2(W; X) : \sum_{\alpha \in \mathcal{J}} q^{|\alpha|} \|\eta_\alpha\|_X^2 < \infty \right\}, \quad q > 1;$$

- The space $L_{2,q}(W; X)$, $0 < q < 1$, which is the closure of $L_2(W; X)$ with respect to the norm

$$\|\eta\|_{L_{2,q}(X)} = \left(\sum_{\alpha \in \mathcal{J}} q^{|\alpha|} \|\eta_\alpha\|_X^2 \right)^{1/2}.$$

It follows that

$$L_{2,q_1}(W; X) \subset L_{2,q_2}(W; X), \quad q_1 > q_2,$$

and, for every $q > 1$,

$$L_{2,q}(W; X) \subset \bigcap_{n>0} \mathbb{D}_2^n(W; X).$$

It is also known [13, Section 1.2] that, for $n = 1, 2, \dots$, the space $\mathbb{D}_2^n(W; X)$ is the domain of \mathbf{D}^n , the n -th power of the Malliavin derivative.

Here is another useful property of the spaces $L_{2,q}(W; X)$.

Proposition 2.1. *If $1 < p < \infty$, and $q > p - 1$, then*

$$L_{2,q}(W; X) \subset L_p(W; X).$$

Proof. Let $\eta \in L_{2,q}(W; X)$. The hypercontractivity property of the Ornstein-Uhlenbeck operator [13, Theorem 1.4.1] implies¹

$$\left(\mathbb{E} \left\| \sum_{|\alpha|=n} \eta_\alpha \xi_\alpha \right\|_X^p \right)^{1/p} \leq (p-1)^{n/2} \left(\sum_{|\alpha|=n} \|\eta_\alpha\|_X^2 \right)^{1/2}.$$

It remains to apply the triangle inequality, followed by the Cauchy-Schwarz inequality:

$$\left(\mathbb{E} \|\eta\|_X^p \right)^{1/p} \leq \sum_{n=0}^{\infty} (p-1)^{n/2} \left(\sum_{|\alpha|=n} \|\eta_\alpha\|_X^2 \right)^{1/2} \leq \left(\sum_{n=0}^{\infty} \left(\frac{p-1}{q} \right)^n \right)^{1/2} \|\eta\|_{L_{2,q}(W; X)}.$$

□

Definition 2.2. *For $\eta \in L_2(W; X)$ and $\zeta \in L_2(W; \mathbb{R})$, the Wick product $\eta \diamond \zeta$ is defined by*

$$(\eta \diamond \zeta)_\alpha = \sum_{\beta, \gamma \in \mathcal{J}: \beta + \gamma = \alpha} \left(\frac{\alpha!}{\beta! \gamma!} \right)^{1/2} \eta_\beta \zeta_\gamma. \quad (2.6)$$

To make sense of $\eta_\beta \zeta_\gamma$, the definition requires at least one of η, ζ to be real-valued. The normalization in (2.6) ensures that, for every n, m, k ,

$$H_n(\xi_k) \diamond H_m(\xi_k) = H_{n+m}(\xi_k),$$

where ξ_k is one of the standard Gaussian random variables (2.3) and H_n is the Hermite polynomial (2.5).

Remark 2.3. *If $\eta \in L_2(W; L_2((0, \pi)))$ and η is adapted, that is, for every $x \in [0, \pi]$, the random variable $\eta(x)$ is measurable with respect to the sigma-algebra generated by $\dot{W}(\chi_{[0,y]})$, $0 \leq y \leq x$, then, by [4, Proposition 2.5.4 and Theorem 2.5.9],*

$$\int_0^x \eta(x) \diamond \dot{W}(x) dx = \int_0^x \eta(x) dW(x),$$

where the right-hand side is the Itô integral with respect to the standard Brownian motion $W(x) = \dot{W}(\chi_{[0,x]})$. This connection with the Itô integral does not help when it comes to equation (1.4): the structure of the heat kernel implies that, for every $x \in (0, \pi)$, the solution $u = u(t, x)$ of (1.4) depends on all of the trajectory of $W(x)$, $x \in (0, \pi)$, and therefore is not adapted as a function of x .

¹In fact, a better reference is the un-numbered equation at the bottom of page 62 in [13].

Given a fixed $\alpha \in \mathcal{J}$, the sum in (2.6) contains finitely many terms, but, in general, $\sum_{\alpha \in \mathcal{J}} \left\| (\eta \diamond \zeta)_\alpha \right\|_X^2 = \infty$ so that $\eta \diamond \zeta$ is not square-integrable.

Here is a sufficient condition for the Wick product to be square-integrable.

Proposition 2.4. *If*

$$\zeta = \sum_k b_k \xi_k, \quad b_k \in \mathbb{R}, \quad (2.7)$$

and $\sum_k b_k^2 < \infty$, then $\eta \mapsto \eta \diamond \zeta$ is a bounded linear operator from $\mathbb{D}_2^1(W; X)$ to $L_2(W; X)$.

Proof. By (2.6),

$$\mathbb{E} \|\eta \diamond \zeta\|_X^2 = \sum_{\alpha \in \mathcal{J}} \left\| \sum_k \sqrt{\alpha_k} b_k \eta_{\alpha^-(k)} \right\|_X^2.$$

By the Cauchy-Schwarz inequality,

$$\left\| \sum_k \sqrt{\alpha_k} b_k \eta_{\alpha^-(k)} \right\|_X^2 \leq |\alpha| \sum_k b_k^2 \|\eta_{\alpha^-(k)}\|_X^2.$$

After summing over all α and shifting the summation index,

$$\mathbb{E} \|\eta \diamond \zeta\|_X^2 \leq \left(\sum_k b_k^2 \right) \sum_{\alpha \in \mathcal{J}} (|\alpha| + 1) \|\eta_\alpha\|_X^2,$$

concluding the proof. \square

Note that, while $\dot{W}(x)$ is of the form (2.7) (cf. (2.4)), Proposition 2.4 does not apply: for a typical value of $x \in [0, \pi]$, $\sum_k |\mathbf{m}_k(x)|^2 = +\infty$. Thus, without either adaptedness of η or square-integrability of \dot{W} , an investigation of the Wick product $\eta \diamond \dot{W}(x)$ requires additional constructions.

One approach (cf. [11]) is to note that if (2.7) is a linear combination of ξ_k , then, by (2.6), the number

$$(\eta \diamond \zeta)_\alpha = \sum_k \sqrt{\alpha_k} b_k \eta_{\alpha^-(k)}$$

is well-defined for every $\alpha \in \mathcal{J}$ regardless of whether the series $\sum_k b_k^2$ converges or diverges. This observation allows an extension of the operation \diamond to spaces much bigger than $L_2(W; X)$ and $L_2(W; \mathbb{R})$; see [11, Proposition 2.7]. In particular, both \dot{W} and $\eta \diamond \dot{W}$, with

$$\left(\eta \diamond \dot{W} \right)_\alpha = \sum_k \sqrt{\alpha_k} \mathbf{m}_k \eta_{\alpha^-(k)}, \quad (2.8)$$

become generalized random elements with values in $L_2((0, \pi))$.

An alternative approach, which we will pursue in this paper, is to consider \dot{W} and $\eta \diamond \dot{W}$ as usual (square integrable) random elements with values in a space of generalized functions.

For $\gamma \in \mathbb{R}$, define the operator

$$\Lambda^\gamma = \left(I - \frac{\partial^2}{\partial x^2} \right)^{\gamma/2} \quad (2.9)$$

on $L_2((0, \pi))$ by

$$(\Lambda^\gamma f)(x) = \sum_{k=1}^{\infty} (1 + (k-1)^2)^{\gamma/2} f_k \mathbf{m}_k(x), \quad (2.10)$$

where, for a smooth f with compact support in $(0, \pi)$,

$$f_k = \int_0^\pi f(x) \mathbf{m}_k(x) dx;$$

recall that $\{\mathbf{m}_k, k \geq 1\}$ is the Fourier cosine basis (2.2) in $L_2((0, \pi))$ so that

$$\Lambda^2 \mathbf{m}_k(x) = \mathbf{m}_k(x) + \mathbf{m}_k''(x) = (1 + (k-1)^2) \mathbf{m}_k(x).$$

If $\gamma > 1/2$, then, by (2.10),

$$(\Lambda^{-\gamma} f)(x) = \int_0^\pi R_\gamma(x, y) f(y) dy, \quad (2.11)$$

where

$$R_\gamma(x, y) = \sum_{k \geq 1} (1 + (k-1)^2)^{-\gamma/2} \mathbf{m}_k(x) \mathbf{m}_k(y). \quad (2.12)$$

Definition 2.5. The Sobolev space $H_2^\gamma((0, \pi))$ is $\Lambda^{-\gamma}(L_2((0, \pi)))$. The norm $\|f\|_\gamma$ in the space is defined by

$$\|f\|_\gamma = \|\Lambda^\gamma f\|_0.$$

The next result is a variation on the theme of Proposition 2.4.

Theorem 2.6. If $\gamma > 1/2$, then $\eta \mapsto \eta \diamond \dot{W}$ is a bounded linear operator from $\mathbb{D}_2^1(W; L_2((0, \pi)))$ to $L_2(W; H_2^{-\gamma}((0, \pi)))$.

Proof. By (2.8), if

$$\eta = \sum_{\alpha \in \mathcal{J}} \eta_\alpha \xi_\alpha$$

with $\eta_\alpha \in L_2((0, \pi))$, then

$$(\eta \diamond \dot{W})(x) = \sum_{\alpha \in \mathcal{J}} \left(\sum_k \sqrt{\alpha_k} \mathbf{m}_k(x) \eta_{\alpha^-(k)}(x) \right) \xi_\alpha,$$

so that

$$\mathbb{E} \|\eta \diamond \dot{W}\|_{-\gamma}^2 = \sum_{\alpha \in \mathcal{J}} \left\| \sum_k \sqrt{\alpha_k} \Lambda^{-\gamma}(\mathbf{m}_k \eta_{\alpha^-(k)}) \right\|_0^2.$$

By the Cauchy-Schwarz inequality,

$$\left\| \sum_k \sqrt{\alpha_k} \Lambda^{-\gamma}(\mathbf{m}_k \eta_{\alpha^-(k)}) \right\|_0^2 \leq |\alpha| \sum_k \int_0^\pi \left(\Lambda^{-\gamma}(\mathbf{m}_k \eta_{\alpha^-(k)}) \right)^2(x) dx.$$

After summing over all α and shifting the summation index,

$$\mathbb{E}\|\eta \diamond \dot{W}\|_{-\gamma}^2 \leq \sum_{\alpha \in \mathcal{J}} (|\alpha| + 1) \sum_k \int_0^\pi \left(\Lambda^{-\gamma}(\mathbf{m}_k \eta_\alpha) \right)^2(x) dx.$$

By (2.11) and Parseval's equality,

$$\sum_k \int_0^\pi \left(\Lambda^{-\gamma}(\mathbf{m}_k \eta_\alpha) \right)^2(x) dx = \int_0^\pi \int_0^\pi R_\gamma^2(x, y) \eta_\alpha^2(y) dy dx,$$

and then (2.12) implies

$$\int_0^\pi R_\gamma^2(x, y) dx = \sum_{k \geq 1} (1 + (k-1)^2)^{-\gamma} \mathbf{m}_k^2(y) \leq \frac{2}{\pi} \sum_{k \geq 0} \frac{1}{(1 + k^2)^\gamma},$$

that is,

$$\int_0^\pi \int_0^\pi R_\gamma^2(x, y) \eta_\alpha^2(y) dy dx \leq C_\gamma \|\eta_\alpha\|_0^2, \quad C_\gamma = \frac{2}{\pi} \sum_{k \geq 0} \frac{1}{(1 + k^2)^\gamma}.$$

As a result,

$$\mathbb{E}\|\eta \diamond \dot{W}\|_{-\gamma}^2 \leq C_\gamma \sum_{\alpha \in \mathcal{J}} (|\alpha| + 1) \|\eta_\alpha\|_0^2,$$

concluding the proof of Theorem 2.6. \square

3. THE CHAOS SOLUTION

Let (V, H, V') be a normal triple of Hilbert spaces, that is

- $V \subset H \subset V'$ and the embeddings $V \subset H$ and $H \subset V'$ are dense and continuous;
- The space V' is dual to V relative to the inner product in H ;
- There exists a constant $C_H > 0$ such that $|(u, v)_H| \leq C_H \|u\|_V \|v\|_{V'}$, for all $u \in V$ and $v \in H$.

An abstract homogeneous Wick-Itô-Skorohod evolution equation in (V, H, V') , driven by the collection $\{\xi_k, k \geq 1\}$ of iid standard Gaussian random variables, is

$$\dot{u}(t) = Au(t) + \sum_k M_k u(t) \diamond \xi_k, \quad t > 0, \quad (3.1)$$

where A and M_k are bounded linear operators from V to V' . Except for Section 8, everywhere else in the paper, the initial condition $u(0) \in H$ is non-random.

Definition 3.1. *The chaos solution of (3.1) is the collection of functions $\{u_\alpha = u_\alpha(t), t > 0, \alpha \in \mathcal{J}\}$ satisfying the propagator*

$$\begin{aligned} \dot{u}_{(0)}(t) &= Au_{(0)}, \quad u_{(0)}(0) = u(0), \\ u_\alpha &= Au_\alpha + \sum_k \sqrt{\alpha_k} M_k u_{\alpha - (k)}, \quad u_\alpha(0) = 0, \quad |\alpha| > 0. \end{aligned}$$

It is known [11, Theorem 3.10] that if the deterministic equation $\dot{v} = Av$ is well-posed in (V, H, V') , then (3.1) has a unique chaos solution

$$u_{\alpha}(t) = \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} \Phi_{t-s_n} M_{k_{\sigma(n)}} \cdots \Phi_{s_2-s_1} M_{k_{\sigma(1)}} \Phi_{s_1} u_0 ds_1 \cdots ds_n, \quad (3.2)$$

where

- \mathcal{P}_n is the permutation group of the set $(1, \dots, n)$;
- $K_{\alpha} = \{k_1, \dots, k_n\}$ is the characteristic set of α ;
- Φ_t is the semigroup generated by A : $u_{(0)}(t) = \Phi_t u_0$.

Once constructed, the chaos solution does not depend on the particular choice of the basis in $L_2(W; H)$ [11, Theorem 3.5]. In general, though,

$$\sum_{\alpha \in \mathcal{J}} \|u_{\alpha}(t)\|_H^2 = \infty,$$

that is, the chaos solution belongs to a space that is bigger than $L_2(W; H)$; cf. [11, Remark 3.14].

On the one hand, equation (1.4) is a particular case of (3.1): $Af(x) = f''(x)$ with zero Neumann boundary conditions, $M_k f(x) = \mathbf{m}_k(x)f(x)$, $H = L_2((0, \pi))$, $V = H^1((0, \pi))$, $V' = H^{-1}((0, \pi))$. The corresponding propagator becomes

$$\begin{aligned} \frac{\partial u_{(0)}(t, x)}{\partial t} &= \frac{\partial^2 u_{(0)}(t, x)}{\partial x^2}, \quad u_{(0)}(0, x) = u_0(x), \\ \frac{\partial u_{\alpha}(t, x)}{\partial t} &= \frac{\partial^2 u_{\alpha}(t, x)}{\partial x^2} + \sum_k \sqrt{\alpha_k} \mathbf{m}_k(x) u_{\alpha - (k)}(t, x), \quad u_{\alpha}(0, x) = 0, \quad |\alpha| > 0. \end{aligned} \quad (3.3)$$

Then existence and uniqueness of the chaos solution of (1.4) are immediate:

Proposition 3.2. *If $u_0 \in L_2((0, \pi))$, then equation (1.4), considered in the normal triple $(H^1((0, \pi)), L_2((0, \pi)), H^{-1}((0, \pi)))$, has a unique chaos solution.*

Proof. This follows from [11, Theorems 3.10]. □

On the other hand, equation (1.4) has two important features that are, in general, not present in (3.1):

- The semigroup Φ_t has a kernel $\mathbf{p}(t, x, y)$:

$$\Phi_t f(x) = \int_0^{\pi} \mathbf{p}(t, x, y) f(y) dy, \quad t > 0, \quad (3.4)$$

where

$$\mathbf{p}(t, x, y) = \sum_{k \geq 1} e^{-(k-1)^2 t} \mathbf{m}_k(x) \mathbf{m}_k(y) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{k=1}^{\infty} e^{-k^2 t} \cos(kx) \cos(ky). \quad (3.5)$$

- By Parseval's equality,

$$\sum_k \left(\int_0^\pi f(x) \mathbf{m}_k(x) dx \right)^2 = \int_0^\pi f^2(x) dx. \quad (3.6)$$

In fact, the properties of the chaos solution of (1.4) are closely connected with the properties of the function $\mathbf{p}(t, x, y)$ from (3.5). Below are some of the properties we will need.

Proposition 3.3. *For $t > 0$ and $x, y \in [0, \pi]$,*

$$\begin{aligned} 0 \leq \mathbf{p}(t, x, y) &\leq \frac{\sqrt{t} + 1}{\sqrt{t}}, \\ |\mathbf{p}_x(t, x, y)| &\leq \frac{4}{t}, \quad |\mathbf{p}_{xx}(t, x, y)| \leq \frac{27}{t^{3/2}}, \quad |\mathbf{p}_t(t, x, y)| \leq \frac{27}{t^{3/2}}. \end{aligned} \quad (3.7)$$

Proof. The maximum principle implies $0 \leq \mathbf{p}(t, x, y)$. To derive other inequalities, note that, by integral comparison,

$$\sum_{k \geq 1} e^{-k^2 t} \leq \int_0^\infty e^{-x^2 t} dx = \frac{\sqrt{\pi}}{2\sqrt{t}}, \quad t > 0,$$

and more generally, for $t > 0$, $r \geq 1$,

$$\sum_{k \geq 1} k^r e^{-k^2 t} \leq \left(\frac{r}{2t} \right)^{(r+1)/2} + \int_0^\infty x^r e^{-x^2 t} dx \leq \frac{(r+1)^{(r+1)}}{t^{(r+1)/2}}. \quad (3.8)$$

To complete the proof, we use

$$\begin{aligned} |\mathbf{p}(t, x, y)| &\leq \frac{1}{2} + \frac{2}{\pi} \sum_{k \geq 1} e^{-k^2 t}, \quad |\mathbf{p}_x(t, x, y)| \leq \sum_{k \geq 1} k e^{-k^2 t}, \\ |\mathbf{p}_{xx}(t, x, y)| &\leq \sum_{k \geq 1} k^2 e^{-k^2 t}, \quad |\mathbf{p}_t(t, x, y)| \leq \sum_{k \geq 1} k^2 e^{-k^2 t}. \end{aligned}$$

□

The main consequence of (3.4) and (3.6) is

Proposition 3.4. (1) *For $|\alpha| = 0$,*

$$\|u_{(\mathbf{0})}(t, \cdot)\|_0 \leq \|u_0\|_0, \quad t > 0, \quad (3.9)$$

and

$$|u_{(\mathbf{0})}(s, y)| \leq C(p, s, t) \|u_0\|_{L_p((0, \pi))}, \quad 0 < s \leq t, \quad 0 \leq y \leq \pi, \quad (3.10)$$

with

$$C(p, s, t) = \begin{cases} (1 + \sqrt{t}) s^{-1/2}, & \text{if } p = 1, \\ \pi^{1/p'} (1 + \sqrt{t}) s^{-1/2}, & \text{if } 1 < p < +\infty, \quad p' = \frac{p}{p-1}, \\ 1, & \text{if } p = +\infty. \end{cases}$$

In particular,

$$C(p, s, t) \leq \pi(1 + \sqrt{t})s^{-1/2} \quad (3.11)$$

for all $0 < s \leq t$ and $1 \leq p \leq +\infty$.

(2) For $|\alpha| = n \geq 1$,

$$\begin{aligned} & \sum_{|\alpha|=n} |u_\alpha(t, x)|^2 \\ & \leq n! \int_{(0, \pi)^n} \left(\int_{\mathbb{T}_{0,t}^n} \mathbf{p}(t - s_n, x, y_n) \cdots \mathbf{p}(s_2 - s_1, y_2, y_1) u_{(\mathbf{0})}(s_1, y_1) ds^n \right)^2 dy^n. \end{aligned} \quad (3.12)$$

Proof. (1) For $|\alpha| = 0$,

$$u_{(\mathbf{0})}(s, y) = \int_0^\pi \mathbf{p}(s, y, z) u_0(z) dz,$$

with \mathbf{p} from (3.5). Then

$$\|u_{(\mathbf{0})}(s, \cdot)\|_0 = \sum_{k \geq 1} e^{-(k-1)^2 s} u_{0,k}^2,$$

from which (3.9) follows.

To derive (3.10) when $p < \infty$, we use the Hölder inequality and (3.7); if $p = +\infty$, then we use $\int_0^\pi \mathbf{p}(s, y, z) dz = 1$ instead of the upper bound in (3.7).

(2) It follows from (3.2) that, for $|\alpha| \geq 1$,

$$\begin{aligned} u_\alpha(t, x) &= \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_{(0, \pi)^n} \int_{\mathbb{T}_{0,t}^n} \mathbf{p}(t - s_n, x, y_n) \mathbf{m}_{k_{\sigma(n)}}(y_n) \\ & \quad \cdots \mathbf{p}(s_2 - s_1, y_2, y_1) \mathbf{m}_{k_{\sigma(1)}}(y_1) u_{(\mathbf{0})}(s_1, y_1) ds^n dy^n. \end{aligned} \quad (3.13)$$

Using (3.4) and notations

$$\mathbf{e}_\alpha(y_1, \dots, y_n) = \frac{1}{\sqrt{n! \alpha!}} \sum_{\sigma \in \mathcal{P}_n} \mathbf{m}_{k_{\sigma(n)}}(y_n) \cdots \mathbf{m}_{k_{\sigma(1)}}(y_1),$$

$$F_n(t, x; y_1, \dots, y_n) = \int_{\mathbb{T}_{0,t}^n} \mathbf{p}(t - s_n, x, y_n) \cdots \mathbf{p}(s_2 - s_1, y_2, y_1) u_{(\mathbf{0})}(s_1, y_1) ds^n, \quad (3.14)$$

we re-write (3.13) as

$$u_\alpha(t, x) = \sqrt{n!} \int_{(0, \pi)^n} F_n(t, x; y_1, \dots, y_n) \mathbf{e}_\alpha(y_1, \dots, y_n) dy^n. \quad (3.15)$$

The collection $\{\mathbf{e}_\alpha, |\alpha| = n\}$ is an orthonormal basis in the symmetric part of the space $L_2((0, \pi)^n)$, so that u_α becomes the corresponding Fourier coefficient of the function F_n , and (3.12) becomes Bessel's inequality. \square

Remark 3.5. It follows from (3.15) that

$$\sum_{|\alpha|=n} |u_\alpha(t, x)|^2 = n! \int_{(0, \pi)^n} \tilde{F}_n^2(t, x; y_1, \dots, y_n) dy^n,$$

where

$$\tilde{F}_n(t, x; y_1, \dots, y_n) = \frac{1}{n!} \sum_{\sigma \in \mathcal{P}_n} F_n(t, x; y_{\sigma(1)}, \dots, y_{\sigma(n)})$$

is the symmetrization of F_n from (3.14). By the Cauchy-Schwarz inequality,

$$\|\tilde{F}_n\|_{L_2((0, \pi)^n)} \leq \|F_n\|_{L_2((0, \pi)^n)},$$

and a separate analysis is necessary to establish a more precise connection between $\|\tilde{F}_n\|_{L_2((0, \pi)^n)}$ and $\|F_n\|_{L_2((0, \pi)^n)}$. The upper bound (3.12) is enough for the purposes of this paper.

4. BASIC REGULARITY OF THE CHAOS SOLUTION

The objective of this section is to show that, for each $t > 0$, the chaos solution of (1.4) is a regular, as opposed to generalized, random variable, and to introduce the main techniques necessary to establish better regularity of the solution.

Theorem 4.1. *If $u_0 \in L_2((0, \pi))$, then, for every $t > 0$, the solution of (1.4) satisfies*

$$u(t, \cdot) \in \bigcap_{q>1} L_{2,q}(W; L_2((0, \pi))). \quad (4.1)$$

Proof. It follows from (3.12) that

$$\begin{aligned} & \sum_{|\alpha|=n} |u_\alpha(t, x)|^2 \\ & \leq n! \int_{(0, \pi)^n} \int_{\mathbb{T}_{0,t}^n} \int_{\mathbb{T}_{0,t}^n} \left(\mathbf{p}(t - s_n, x, y_n) \cdots \mathbf{p}(s_2 - s_1, y_2, y_1) u_{(0)}(s_1, y_1) \right. \\ & \quad \left. \times \mathbf{p}(t - r_n, x, y_n) \cdots \mathbf{p}(r_2 - r_1, y_2, y_1) u_{(0)}(r_1, y_1) \right) ds^n dr^n dy^n. \end{aligned} \quad (4.2)$$

We now integrate both sides of (4.2) with respect to x and use the semigroup property

$$\int_0^\pi \mathbf{p}(t, x, y) \mathbf{p}(s, y, z) dy = \mathbf{p}(t + s, x, z) \quad (4.3)$$

together with (3.7) to evaluate the integrals over $(0, \pi)$ on the right-hand side, starting from the outer-most integral. We also use (3.9). The result is

$$\begin{aligned} \sum_{|\alpha|=n} \|u_\alpha(t, \cdot)\|_0^2 & \leq n! \|u_0\|_0^2 (1 + \sqrt{t})^{2n} \int_{\mathbb{T}_{0,t}^n} \int_{\mathbb{T}_{0,t}^n} (2t - s_n - r_n)^{-1/2} \\ & \quad (s_n + r_n - s_{n-1} - r_{n-1})^{-1/2} \cdots (s_2 + r_2 - s_1 - r_1)^{-1/2} ds^n dr^n. \end{aligned} \quad (4.4)$$

Next, we use the inequality $4pq \leq (p + q)^2$, $p, q > 0$, to find

$$(p + q)^{-1/2} \leq p^{-1/4} q^{-1/4}, \quad (4.5)$$

so that

$$\begin{aligned} & \int_{\mathbb{T}_{0,t}^n} \int_{\mathbb{T}_{0,t}^n} (2t - s_n - r_n)^{-1/2} (s_n + r_n - s_{n-1} - r_{n-1})^{-1/2} \cdots (s_2 + r_2 - s_1 - r_1)^{-1/2} ds^n dr^n \\ & \leq \left(\int_{\mathbb{T}_{0,t}^n} (t - s_n)^{-1/4} (s_n - s_{n-1})^{-1/4} \cdots (s_2 - s_1)^{-1/4} ds^n \right)^2 = \left(\frac{(\Gamma(3/4))^n}{\Gamma((3/4)n + 1)} \right)^2 t^{3n/2}, \end{aligned} \quad (4.6)$$

where Γ is the Gamma function

$$\Gamma(y) = \int_0^\infty t^{y-1} e^{-t} dt.$$

The last equality in (4.6) follows by induction using

$$\int_0^t s^p (t-s)^q ds = t^{p+q+1} \frac{\Gamma(1+p)\Gamma(1+q)}{\Gamma(2+p+q)}, \quad p, q > -1. \quad (4.7)$$

Combining (4.2), (4.4), and (4.6),

$$\sum_{|\alpha|=n} \|u_\alpha(t, \cdot)\|_0^2 \leq n! \left(\frac{(\Gamma(3/4))^n}{\Gamma((3/4)n + 1)} \right)^2 (1 + \sqrt{t})^{2n} t^{3n/2} \|u_0\|_0^2.$$

As a consequence of the Stirling formula,

$$\Gamma(1+p) \geq \sqrt{2\pi p} p^p e^{-p} \quad \text{and} \quad n! \leq 2\sqrt{\pi n} n^n e^{-n},$$

meaning that

$$\sum_{|\alpha|=n} \|u_\alpha(t, \cdot)\|_0^2 \leq C^n(t) n^{-n/2} \|u_0\|_0^2, \quad t > 0, \quad (4.8)$$

with

$$C(t) = (4/3)^{3/2} e^{1/2} \Gamma^2(3/4) (1 + \sqrt{t})^2 t^{3/2}.$$

Since

$$\mathbb{E} \|u(t, \cdot)\|_{L_{2,q}(L_2((0,\pi)))}^2 = \sum_{n=0}^\infty q^n \sum_{\alpha \in \mathcal{J}: |\alpha|=n} \|u_\alpha(t, \cdot)\|_0^2,$$

and the series

$$\sum_{n \geq 1} \frac{C^n}{n^{n/2}} = \sum_{n \geq 1} \left(\frac{C}{\sqrt{n}} \right)^n$$

converges for every $C > 1$, we get (4.1) and conclude the proof of Theorem 4.1. \square

Corollary 4.2. *If $u_0 \in L_2((0, \pi))$, then the chaos solution is an $L_2((0, \pi))$ -valued random process and, for all $t \geq 0$,*

$$\mathbb{E} \|u(t, \cdot)\|_0^p < \infty, \quad 1 \leq p < \infty.$$

Proof. This follows from (4.1) and Proposition 2.1. \square

We will need a slightly more general family of integrals than the one appearing on the right-hand side of (4.6):

$$\begin{aligned} I_1(t; \alpha, \beta) &= \int_0^t (t-s)^{-\alpha} s^{-\beta} ds, \\ I_n(t; \alpha, \beta) &= \int_{\mathbb{T}_{0,t}^n} (t-s_n)^{-\alpha} \prod_{k=2}^n (s_k - s_{k-1})^{-1/4} s_1^{-\beta} ds^n, \quad n = 2, 3, \dots, \end{aligned} \quad (4.9)$$

for $\alpha \in (0, 1)$, $\beta \in [0, 1)$. Note that

$$I_1(t; \alpha, \beta) = \int_0^t (t-s)^{-\alpha} s^{-\beta} ds = \frac{\Gamma(1-\alpha)\Gamma(1-\beta)}{\Gamma(2-\alpha-\beta)} t^{1-\alpha-\beta}$$

and

$$I_n(t; \alpha, \beta) = \int_0^t (t-s_n)^{-\alpha} I_{n-1}(s_n; 1/4, \beta) ds_n, \quad n \geq 1.$$

By induction and (4.7),

$$I_n(t; \alpha, \beta) = \frac{(\Gamma(3/4))^{n-1} \Gamma(1-\alpha) \Gamma(1-\beta)}{\Gamma((3n+5-4\alpha-4\beta)/4)} t^{(3n+1-4\alpha-4\beta)/4},$$

and then

$$n! I_n^2(t; \alpha, \beta) \leq C^n(\alpha, \beta, t) n^{-n/2}; \quad (4.10)$$

cf. (4.8).

Next, we show that the chaos solution of (1.4) is, in fact, a **random field solution**, that is, $u(t, x)$ is well-defined as a random variable for every $t > 0$, $x \in [0, \pi]$.

Theorem 4.3. *If $u_0 \in L_p((0, \pi))$ for some $1 \leq p \leq \infty$, then, for every $t > 0$ and $x \in [0, \pi]$,*

$$u(t, x) \in \bigcap_{q>1} L_{2,q}(W; \mathbb{R}). \quad (4.11)$$

Proof. By Proposition 3.4, inequality (4.2) becomes

$$\begin{aligned} \sum_{|\alpha|=n} |u_\alpha(t, x)|^2 &\leq n! \pi^2 (1 + \sqrt{t})^2 \|u_0\|_{L_p((0, \pi))}^2 \\ &\int_{(0, \pi)^n} \iint_{\mathbb{T}_{0,t}^n \times \mathbb{T}_{0,t}^n} \left(\mathbf{p}(t - s_n, x, y_n) \cdots \mathbf{p}(s_2 - s_1, y_2, y_1) s_1^{-1/2} \right. \\ &\quad \left. \times \mathbf{p}(t - r_n, x, y_n) \cdots \mathbf{p}(r_2 - r_1, y_2, y_1) r_1^{-1/2} \right) ds^n dr^n dy^n. \end{aligned} \quad (4.12)$$

We now use the semigroup property (4.3) together with (3.7) to evaluate the integrals over $(0, \pi)$ on the right-hand side of (4.12) starting from the inner-most integral with

respect to y_1 . The result is

$$\sum_{|\alpha|=n} |u_\alpha(t, x)|^2 \leq n! \pi^2 (1 + \sqrt{t})^{2(n+1)} \|u_0\|_{L_p((0, \pi))}^2 \iint_{\mathbb{T}_{0,t}^n \times \mathbb{T}_{0,t}^n} (2t - s_n - r_n)^{-1/2} (s_n + r_n - s_{n-1} - r_{n-1})^{-1/2} \cdots (s_2 + r_2 - s_1 - r_1)^{-1/2} s_1^{-1/2} r_1^{-1/2} ds^n dr^n. \quad (4.13)$$

Next, similar to (4.6), we use (4.5) and (4.9) to compute

$$\begin{aligned} & \iint_{\mathbb{T}_{0,t}^n \times \mathbb{T}_{0,t}^n} (2t - s_n - r_n)^{-1/2} (s_n + r_n - s_{n-1} - r_{n-1})^{-1/2} \\ & \quad \cdots (s_2 + r_2 - s_1 - r_1)^{-1/2} s_1^{-1/2} r_1^{-1/2} ds^n dr^n \\ & \leq \left(\int_{\mathbb{T}_{0,t}^n} (t - s_n)^{-1/4} (s_n - s_{n-1})^{-1/4} \cdots (s_2 - s_1)^{-1/4} s_1^{-1/2} ds^n \right)^2 \\ & = I_n^2(t; 1/4, 1/2). \end{aligned} \quad (4.14)$$

Combining (4.13) with (4.14) and (4.10),

$$\sum_{|\alpha|=n} |u_\alpha(t, x)|^2 \leq C^n(t) n^{-n/2} \|u_0\|_{L_p((0, \pi))}^2, \quad (4.15)$$

for a suitable $C(t)$. Then (4.15) leads to (4.11) in the same way as (4.8) lead to (4.1), completing the proof of Theorem 4.3. □

Corollary 4.4. *For every $t > 0$, $x \in [0, \pi]$, and $1 \leq p < \infty$,*

$$\mathbb{E}|u(t, x)|^p < \infty.$$

Proof. This follows from (4.11) and Proposition 2.1. □

Finally, we establish a version of the maximum principle for the chaos solution.

Theorem 4.5. *If $u_0(x) \geq 0$ for all $x \in [0, \pi]$, and $u = u(t, x)$ is a random field solution of (1.4) such that*

$$u \in L_2(\Omega \times [0, T], L_p((0, \pi))),$$

then, with probability one, $u(t, x) \geq 0$ for all $t \in [0, T]$ and $x \in [0, \pi]$.

Proof. Let $h = h(x)$ be a smooth function with compact support in $(0, \pi)$ and define

$$V(t, x; h) = \mathbb{E} \left(u(t, x) \exp \left(\dot{W}(h) - \frac{1}{2} \|h\|_{L_2(0, \pi)}^2 \right) \right).$$

Writing $h(x) = \sum_{k=1}^{\infty} h_k \mathbf{m}_k(x)$ and $h^\alpha = \prod_k h_k^{\alpha_k}$, we find

$$V(t, x; h) = \sum_{\alpha \in \mathcal{J}} \frac{h^\alpha u_\alpha(t, x)}{\sqrt{\alpha!}}.$$

By (3.3), the function $V = V(t, x; h)$ satisfies

$$\frac{\partial V(t, x; h)}{\partial t} = \frac{\partial^2 V(t, x; h)}{\partial x^2} + h(x)V(t, x; h), \quad 0 < t \leq T, \quad x \in (0, \pi),$$

with $V(0, x; h) = u_0(x)$ and $V_x(t, 0; h) = V_x(t, \pi; h) = 0$, and then the maximum principle implies $V(t, x; h) \geq 0$ for all $t \in [0, T]$, $x \in [0, \pi]$. The conclusion of the theorem now follows, because the collection of the random variables

$$\left\{ \exp \left(\dot{W}(h) - \frac{1}{2} \|h\|_{L_2((0, \pi))}^2 \right), \quad h \text{ smooth with compact support in } (0, \pi) \right\}$$

is dense in $L_2(W; \mathbb{R})$; cf. [14, Lemma 4.3.2]. \square

Remark 4.6. *If $u = u(t, x)$ is continuous in (t, x) , then there exists a single probability-one subset Ω' of Ω such that $u = u(t, x, \omega) > 0$ for all $t \in [0, T]$, $x \in [0, \pi]$, and $\omega \in \Omega'$.*

5. EQUATION WITH ADDITIVE NOISE

The objective of this section is to establish the bench-mark space-time regularity result for (1.4) by considering the corresponding equation with additive noise:

$$\begin{aligned} U_t &= U_{xx} + \dot{W}(x), \quad t > 0, \quad x \in (0, \pi), \\ U(0, x) &= 0, \quad U_x(t, 0) = U_x(t, \pi) = 0. \end{aligned} \tag{5.1}$$

By the variation of parameters formula, the solution of (5.1) is

$$U(t, x) = \int_0^t \int_0^\pi \mathbf{p}(s, x, y) dW(y) ds.$$

Using (3.5),

$$U(t, x) = \frac{t}{\pi} \zeta_0 + \frac{2}{\pi} \sum_{k \geq 1} k^{-2} (1 - e^{-k^2 t}) \cos(kx) \zeta_k, \tag{5.2}$$

$$U_x(t, x) = -\frac{2}{\pi} \sum_{k \geq 1} k^{-1} (1 - e^{-k^2 t}) \sin(kx) \zeta_k, \tag{5.3}$$

where

$$\zeta_0 = W(\pi), \quad \zeta_k = \int_0^\pi \cos(kx) dW(x), \quad k \geq 1,$$

are independent Gaussian random variables with zero mean. In particular, the series on the right-hand sides of (5.2) and (5.3) converge with probability one for every $t > 0$ and $x \in [0, \pi]$.

Let us now recall the necessary definitions of the Hölder spaces. For a function $f = f(x)$, $x \in (x_1, x_2)$, $-\infty < x_1 < x_2 < +\infty$, we write

$$f \in \mathcal{C}^\alpha((x_1, x_2)), \quad 0 < \alpha \leq 1,$$

or, equivalently, f is Hölder(α), if

$$\sup_{x,y \in (x_1, x_2), x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty.$$

Similarly,

$$f \in \mathcal{C}^{1+\alpha}((x_1, x_2))$$

if f is continuously differentiable on $[x_1, x_2]$ and

$$\sup_{x,y \in (x_1, x_2), x \neq y} \frac{|f'(x) - f'(y)|}{|x - y|^\alpha} < \infty.$$

We also write $f \in \mathcal{C}^{\beta-}((x_1, x_2))$, or f is almost Hölder(β), if $f \in \mathcal{C}^{\beta-\varepsilon}((x_1, x_2))$ for every $\varepsilon \in (0, \beta)$.

The main tool for establishing Hölder regularity of random processes is the Kolmogorov continuity criterion:

Theorem 5.1. *Let T be a positive real number and $X = X(t)$, a real-valued random process on $[0, T]$. If there exist numbers $C > 0$, $p > 1$, and $q \geq p$ such that, for all $t, s \in [0, T]$,*

$$\mathbb{E}|X(t) - X(s)|^q \leq C|t - s|^p,$$

then there exists a modification of X with sample trajectories that are almost Hölder($(p-1)/q$).

Proof. See, for example Karatzas and Shreve [7, Theorem 2.2.8]. □

We now apply Theorem 5.1 to the solution of equation (5.1).

Theorem 5.2. *The random field $U = U(t, x)$ defined in (5.2) satisfies*

$$U(\cdot, x) \in \mathcal{C}^{3/4-}((0, T)), \quad x \in [0, \pi], \quad T > 0; \tag{5.4}$$

$$U_x(\cdot, x) \in \mathcal{C}^{1/4-}((0, T)), \quad x \in [0, \pi], \quad T > 0; \tag{5.5}$$

$$U(t, \cdot) \in \mathcal{C}^{3/2-}((0, \pi)), \quad t > 0. \tag{5.6}$$

Proof. For every $t > 0$ and $x, y \in [0, \pi]$, the random variables $\tilde{U}(t, x) = U(t, x) - \zeta_0 t / \pi$ and $U_x(t, x)$ are Gaussian, so that, by Theorem 5.1, statements (5.4), (5.5), and (5.6) will follow from

$$\mathbb{E}|\tilde{U}(t+h, x) - \tilde{U}(t, x)|^2 \leq C(\varepsilon)h^{3/2-\varepsilon}, \quad \varepsilon \in (0, 3/2), \tag{5.7}$$

$$\mathbb{E}|U_x(t+h, x) - U_x(t, x)|^2 \leq C(\varepsilon)h^{1/2-\varepsilon}, \quad \varepsilon \in (0, 1/2), \tag{5.8}$$

$$\mathbb{E}|U_x(t, x+h) - U_x(t, x)|^2 \leq C(\varepsilon)h^{1-\varepsilon}, \quad \varepsilon \in (0, 1), \tag{5.9}$$

respectively, if we use $p = q\delta/2$ with suitable δ and sufficiently large q .

Using (5.2) and (5.3), and keeping in mind that ζ_k , $k \geq 1$, are iid Gaussian with mean zero and variance $\pi/2$,

$$\mathbb{E}|\tilde{U}(t+h, x) - \tilde{U}(t, x)|^2 = \frac{2}{\pi} \sum_{k \geq 1} k^{-4} e^{-2k^2 t} (1 - e^{-k^2 h})^2 \cos^2(kx); \quad (5.10)$$

$$\mathbb{E}|U_x(t+h, x) - U_x(t, x)|^2 = \frac{2}{\pi} \sum_{k \geq 1} k^{-2} e^{-2k^2 t} (1 - e^{-k^2 h})^2 \cos^2(kx); \quad (5.11)$$

$$\mathbb{E}|U_x(t, x+h) - U_x(t, x)|^2 = \frac{2}{\pi} \sum_{k \geq 1} k^{-2} (1 - e^{-k^2 h})^2 (\sin(k(x+h)) - \sin(kx))^2. \quad (5.12)$$

We also use

$$1 - e^{-\theta} \leq \theta^\alpha, \quad 0 < \alpha \leq 1, \quad \theta > 0, \quad (5.13)$$

$$\sin \theta \leq \theta^\alpha, \quad 0 < \alpha \leq 1, \quad \theta > 0. \quad (5.14)$$

Then

- Inequality (5.7) follows from (3.8), (5.10), and (5.13) by taking $\alpha < 3/4$;
- Inequality (5.8) follows from (3.8), (5.11), and (5.13) with $\alpha < 1/4$;
- Inequality (5.9) follows from (3.8), (5.12), and (5.14) with $\alpha < 1/2$.

□

Remark 5.3. *Similar to [8, Theorem 3.3] in the case of space-time white noise, equalities (5.2) and (5.3) imply that, for every $t > 0$, the random field $U(t, x)$ is infinitely differentiable in t , and the random field $U_x(t, x) + B(x)$ is infinitely differentiable in x , where*

$$B(x) = \frac{2}{\pi} \sum_{k \geq 1} k^{-1} \zeta_k \sin(kx)$$

is a Brownian bridge on $[0, \pi]$.

6. TIME REGULARITY OF THE CHAOS SOLUTION

The objective of this section is to show that the chaos solution of (1.4) has a modification that is almost Hölder(3/4) in time. To simplify the presentation, we will not distinguish different modifications of the solution.

Theorem 6.1. *If $u_0 \in \mathcal{C}^{3/2}((0, \pi))$, then the chaos solution of (1.4) satisfies*

$$u(\cdot, x) \in \mathcal{C}^{3/4-}((0, T))$$

for every $T > 0$ and $x \in [0, \pi]$.

Proof. We need to show that, for every $x \in [0, \pi]$, $h \in (0, 1)$, $\varepsilon \in (0, 3/4)$, $t \in (0, T)$, and $p \in (1, +\infty)$,

$$\left(\mathbb{E}|u(t+h, x) - u(t, x)|^p \right)^{1/p} \leq C(p, T, \varepsilon) h^{3/4-\varepsilon}.$$

Then the statement of the theorem will follow Theorem 5.1.

Recall that $u_{(\mathbf{0})}(t, x)$ is the solution of

$$\frac{\partial u_{(\mathbf{0})}(t, x)}{\partial t} = \frac{\partial^2 u_{(\mathbf{0})}(t, x)}{\partial x^2}, \quad u_{(\mathbf{0})}(0, x) = u_0(x),$$

with boundary conditions

$$\frac{\partial u_{(\mathbf{0})}(t, 0)}{\partial x} = \frac{\partial u_{(\mathbf{0})}(t, \pi)}{\partial x} = 0,$$

that is,

$$\frac{\partial u_{(\mathbf{0})}(t, x)}{\partial t} = (1 - \Lambda^2)u_{(\mathbf{0})}(t, x);$$

the operator Λ is defined in (2.9). Applying [9, Theorem 5.3] to equation

$$U_t(t, x) = (1 - \Lambda^2)U(t, x), \quad U(0, x) = \Lambda^{-1}u_0(x),$$

we conclude that, for each $x \in [0, \pi]$,

$$u_{(\mathbf{0})}(\cdot, x) \in \mathcal{C}^{3/4}((0, T)). \quad (6.1)$$

For $n \geq 1$ and $h \in (0, 1)$, similar to (3.12),

$$\begin{aligned} & \sum_{|\alpha|=n} |u_{\alpha}(t+h, x) - u_{\alpha}(t, x)|^2 \\ & \leq n! \int_{(0, \pi)^n} \left(\int_{\mathbb{T}_{0, t+h}^n} \mathbf{p}(t+h-s_n, x, y_n) \cdots \mathbf{p}(s_2-s_1, y_2, y_1) u_{(\mathbf{0})}(s_1, y_1) ds^n \right. \\ & \quad \left. - \int_{\mathbb{T}_{0, t}^n} \mathbf{p}(t-s_n, x, y_n) \cdots \mathbf{p}(s_2-s_1, y_2, y_1) u_{(\mathbf{0})}(s_1, y_1) ds^n \right)^2 dy^n. \end{aligned} \quad (6.2)$$

We add and subtract

$$\int_{\mathbb{T}_{0, t}^n} \mathbf{p}(t+h-s_n, x, y_n) \cdots \mathbf{p}(s_2-s_1, y_2, y_1) u_{(\mathbf{0})}(s_1, y_1) ds^n$$

inside the square on the right-hand side of (6.2), and then use $(p+q)^2 \leq 2p^2 + 2q^2$ to re-write (6.2) as

$$\begin{aligned} & \sum_{|\alpha|=n} |u_{\alpha}(t+h, x) - u_{\alpha}(t, x)|^2 \\ & \leq 2n! \int_{(0, \pi)^n} \left(\int_t^{t+h} \int_{\mathbb{T}_{0, s_n}^{n-1}} \mathbf{p}(t+h-s_n, x, y_n) \cdots \mathbf{p}(s_2-s_1, y_2, y_1) u_{(\mathbf{0})}(s_1, y_1) ds^n \right)^2 dy^n \\ & + 2n! \int_{(0, \pi)^n} \left(\int_{\mathbb{T}_{0, t}^n} \left[\mathbf{p}(t+h-s_n, x, y_n) - \mathbf{p}(t-s_n, x, y_n) \right] \mathbf{p}(s_n-s_{n-1}, y_n, y_{n-1}) \right. \\ & \quad \left. \cdots \mathbf{p}(s_2-s_1, y_2, y_1) u_{(\mathbf{0})}(s_1, y_1) ds^n \right)^2 dy^n. \end{aligned} \quad (6.3)$$

To estimate the first term on the right-hand side of (6.3), we follow computations similar to (4.6) and (4.13), and use

$$\mathbf{p}(t, x, y) \geq 0, \quad \|u_{(\mathbf{0})}(s, \cdot)\|_{L_\infty((0, \pi))} \leq \|u_0\|_{L_\infty((0, \pi))}, \quad s \geq 0, \quad (6.4)$$

as well as (4.10):

$$\begin{aligned} & 2n! \int_{(0, \pi)^n} \left(\int_t^{t+h} \int_{\mathbb{T}_{0, s_n}^{n-1}} \mathbf{p}(t+h-s_n, x, y_n) \cdots \mathbf{p}(s_2-s_1, y_2, y_1) u_{(\mathbf{0})}(s_1, y_1) ds^n \right)^2 dy^n \\ & \leq 2n! \|u_0\|_{L_\infty((0, \pi))}^2 \left(\int_t^{t+h} \int_{\mathbb{T}_{0, s_n}^{n-1}} (t+h-s_n)^{-1/4} (s_n-s_{n-1})^{1/4} \cdots (s_2-s_1)^{-1/4} ds^n \right)^2 dy^n \\ & \leq 2n! \|u_0\|_{L_\infty((0, \pi))}^2 \left(\int_t^{t+h} (t+h-s_n)^{-1/4} I_{n-1}(s_n; 1/4, 1/4) ds_n \right)^2 \\ & \leq \|u_0\|_{L_\infty((0, \pi))}^2 C^n(t) n^{-n/2} h^{3/2}, \end{aligned} \quad (6.5)$$

with a suitable $C(t)$.

To estimate the second term on the right-hand side of (6.3), define

$$\begin{aligned} \mathcal{I}(t, h, s, r, x) &= \int_0^\pi \left(\mathbf{p}(t+h-s, x, y) - \mathbf{p}(t-s, x, y) \right) \\ &\quad \times \left(\mathbf{p}(t+h-r, x, y) - \mathbf{p}(t-r, x, y) \right) dy. \end{aligned}$$

By (3.5),

$$\mathcal{I}(t, h, s, r, x) = \frac{4}{\pi^2} \sum_{k \geq 1} (e^{-k^2 h} - 1)^2 e^{-k^2(t-s) - k^2(t-r)} \cos^2(kx).$$

Using (5.13) and taking $0 < \gamma < 3/4$, we conclude that

$$\mathcal{I}(t, h, s, r, x) \leq h^{2\gamma} \sum_{k \geq 1} k^{4\gamma} e^{-k^2(2t-s-r)}.$$

Then (3.8) implies

$$\mathcal{I}(t, h, s, r, x) \leq (4\gamma + 1)^{4\gamma+1} h^{2\gamma} (2t-s-r)^{-(1/2+2\gamma)}.$$

Note that $1/2 + 2\gamma < 2$.

We now carry out computations similar to (4.13), and use (4.10) and (6.4):

$$\begin{aligned}
& 2n! \int_{(0,\pi)^n} \left(\int_{\mathbb{T}_{0,t}^n} [\mathbf{p}(t+h-s_n, x, y_n) - \mathbf{p}(t-s_n, x, y_n)] \right. \\
& \quad \times \mathbf{p}(s_2-s_1, y_2, y_1) \cdots \mathbf{p}(s_2-s_1, y_2, y_1) u_{(0)}(s_1, y_1) ds^n \Big)^2 dy^n \\
& \leq 2n! \|u_0\|_{L^\infty((0,\pi))}^2 (1+\sqrt{t})^{2n} \\
& \quad \int_{\mathbb{T}_{0,t}^n \times \mathbb{T}_{0,t}^n} \mathcal{I}(t, h, s_n, r_n, x) \prod_{k=1}^{n-2} (s_{k+1} + r_{k+1} - s_k - r_k)^{-1/2} (s_1 + r_1 - 2s)^{-1/2} ds^n dr^n \\
& \leq h^{2\gamma} \|u_0\|_{L^\infty((0,\pi))}^2 \frac{4(4\gamma+1)^{4\gamma+1}}{\pi} (1+\sqrt{t})^{2n} n! I_n^2(t; 1/4+\gamma, 1/4) \\
& \leq \|u_0\|_{L^\infty((0,\pi))}^2 C^n(t) n^{-n/2} h^{2\gamma},
\end{aligned} \tag{6.6}$$

with a suitable $C(t)$.

Combining (6.5) and (6.6),

$$\sum_{|\alpha|=n} |u_\alpha(t+h, x) - u_\alpha(t, x)|^2 \leq h^{3/2-2\varepsilon} \|u_0\|_{L^\infty((0,\pi))}^2 C^n(t, \varepsilon) n^{-n/2}, \tag{6.7}$$

$\varepsilon \in (0, 3/4)$, $n \geq 1$, and then, by (6.1) and Proposition 2.1,

$$\begin{aligned}
\left(\mathbb{E} |u(t+h, x) - u(t, x)|^p \right)^{1/p} & \leq \sum_{n=0}^{\infty} (p-1)^{n/2} \left(\sum_{|\alpha|=n} |u_\alpha(t+h, x) - u_\alpha(t, x)|^2 \right)^{1/2} \\
& \leq C(p, T, \varepsilon) \|u_0\|_{L^\infty((0,\pi))} h^{3/4-\varepsilon},
\end{aligned}$$

for all $1 < p < +\infty$, $t \in (0, T)$, $\varepsilon \in (0, 3/4)$, $0 < h < 1$, completing the proof of Theorem 6.1. □

7. SPACE REGULARITY OF THE CHAOS SOLUTION

The objective of this section is to show that, for every $t > 0$, the chaos solution

$$u(t, x) = \sum_{\alpha \in \mathcal{J}} u_\alpha(t, x) \xi_\alpha$$

of (1.4) has a modification that is in $\mathcal{C}^{3/2-}((0, \pi))$. As in the previous section, we will not distinguish between different modifications of u .

To streamline the presentation, we will break the argument in two parts: existence of u_x as a random field, followed by Hölder(1/2−) regularity of u_x in space.

Define

$$v(t, x) = \sum_{\alpha \in \mathcal{J}} v_\alpha(t, x) \xi_\alpha,$$

where

$$\begin{aligned} v_{\alpha}(t, x) &= \frac{\partial u_{\alpha}(t, x)}{\partial x} \\ &= \frac{1}{\sqrt{\alpha!}} \int_{(0, \pi)^n} \sum_{\sigma \in \mathcal{P}_n} \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} \mathbf{p}_x(t - s_n, x, y_n) \mathbf{m}_{k_{\sigma}(n)}(y_n) \\ &\quad \times \mathbf{p}(s_n - s_{n-1}, y_n, y_{n-1}) \mathbf{m}_{k_{\sigma}(n-1)} \cdots \mathbf{p}(s_2 - s_1, y_2, y_1) \mathbf{m}_{k_{\sigma}(1)}(y_1) u_0(s_1, y_1) ds^n dy^n. \end{aligned}$$

Theorem 7.1. *Assume that $u_0 \in L_p((0, \pi))$ for some $1 \leq p \leq \infty$. Then, for every $t > 0$ and $x \in (0, \pi)$,*

$$u_x(t, x) \in \bigcap_{q>1} L_{2,q}(W; \mathbb{R}).$$

Proof. By construction, $v = u_x$ as generalized processes. It remains to show that

$$v(t, x) \in \bigcap_{q>1} L_{2,q}(W; \mathbb{R}). \quad (7.1)$$

Similar to (3.12),

$$\begin{aligned} \sum_{|\alpha|=n} |v_{\alpha}(t, x)|^2 &\leq n! \int_{(0, \pi)^n} \left(\int_{\mathbb{T}_{0,t}^n} \mathbf{p}_x(t - s_n, x, y_n) \mathbf{p}(s_n - s_{n-1}, y_n, y_{n-1}) \right. \\ &\quad \left. \cdots \mathbf{p}(s_2 - s_1, y_2, y_1) u_{(0)}(s_1, y_1) ds^n \right)^2 dy^n. \end{aligned} \quad (7.2)$$

Using (3.11),

$$\begin{aligned} &\sum_{|\alpha|=n} |v_{\alpha}(t, x)|^2 \\ &\leq n! \|u_0\|_{L_p((0, \pi))}^2 \pi^2 (1 + \sqrt{t})^2 \int_{(0, \pi)^n} \left(\int_{\mathbb{T}_{0,t}^n} \mathbf{p}_x(t - s_n, x, y_n) \mathbf{p}(s_n - s_{n-1}, y_n, y_{n-1}) \right. \\ &\quad \left. \cdots \mathbf{p}(s_2 - s_1, y_2, y_1) s_1^{-1/2} ds^n \right)^2 dy^n. \end{aligned}$$

By (3.5),

$$\int_0^{\pi} \mathbf{p}_x(t, x, y) \mathbf{p}_x(s, x, y) dy = \frac{4}{\pi^2} \sum_{k=1}^{\infty} k^2 e^{-k^2(t+s)} \sin^2(kx) \leq \frac{27}{(t+s)^{3/2}},$$

and then

$$\begin{aligned}
\sum_{|\alpha|=n} |v_\alpha(t, x)|^2 &\leq 27n!\pi^2(1 + \sqrt{t})^{2n} \|u_0\|_{L_p((0, \pi))}^2 \\
&\times \left(\int_{\mathbb{T}_{0,t}^n} (t - s_{n-1})^{-3/4} (s_{n-1} - s_{n-2})^{-1/4} \cdots (s_2 - s_1)^{-1/4} s_1^{-1/2} ds^n \right)^2 ds \\
&= 27\pi^2(1 + \sqrt{t})^{2n} \|u_0\|_{L_p((0, \pi))}^2 n! I_n^2(t; 3/4, 1/2) \leq \|u_0\|_{L_p((0, \pi))}^2 C^n(t) n^{-n/2}
\end{aligned}$$

with a suitable $C(t)$; cf. (4.10). Then (7.1) follows in the same way as (4.1) followed from (4.8). \square

Remark 7.2. *Similar to the proof of Theorem 6.1, an interested reader can confirm that $u_x(\cdot, x) \in \mathcal{C}^{1/4-}([\delta, T])$ for every $x \in [0, \pi]$ and $T > \delta > 0$.*

Theorem 7.3. *If $u_0 \in L_p((0, \pi))$ for some $1 \leq p \leq \infty$, then, for every $t > 0$,*

$$u_x(t, \cdot) \in \mathcal{C}^{1/2-}((0, \pi)).$$

Proof. We continue to use the notation $v = u_x$. Then the objective is to show that, for every sufficiently small $h > 0$ and every $x \in (0, \pi)$, $t > 0$, $p > 1$, and $\gamma \in (0, 1/2)$,

$$\left(\mathbb{E} |v(t, x+h) - v(t, x)|^p \right)^{1/p} \leq C(t, p, \gamma) h^\gamma; \quad (7.3)$$

then the conclusion of the theorem will follow from the Kolmogorov continuity criterion.

Similar to (3.12),

$$\begin{aligned}
&\sum_{|\alpha|=n} |v_\alpha(t, x+h) - v_\alpha(t, x)|^2 \\
&\leq n! \int_{(0, \pi)^n} \left(\int_{\mathbb{T}_{0,t}^n} [\mathbf{p}_x(t - s_n, x+h, y_n) - \mathbf{p}_x(t - s_n, x, y_n)] \right. \\
&\quad \times \mathbf{p}(s_n - s_{n-1}, y_n, y_{n-1}) \cdots \mathbf{p}(s_2 - s_1, y_2, y_1) u_{(\mathbf{0})}(s_1, y_1) ds^n \Big)^2 dy^n,
\end{aligned}$$

and then

$$\begin{aligned}
&\sum_{|\alpha|=n} |v_\alpha(t, x+h) - v_\alpha(t, x)|^2 \\
&\leq n!\pi^2(1 + \sqrt{t})^2 \|u_0\|_{L_p((0, \pi))}^2 \int_{(0, \pi)^n} \left(\int_{\mathbb{T}_{0,t}^n} [\mathbf{p}_x(t - s_{n-1}, x+h, y_n) - \mathbf{p}_x(t - s_{n-1}, x, y_n)] \right. \\
&\quad \times \mathbf{p}(s_{n-1} - s_{n-2}, y_n, y_{n-1}) \cdots \mathbf{p}(s_1 - s, y_2, y_1) s_1^{-1/2} ds^n \Big)^2 dy^n;
\end{aligned} \quad (7.4)$$

cf. (7.2).

Next, define

$$\begin{aligned} J(t, s, r, x, y, h) \\ = \int_0^\pi (\mathfrak{p}_x(t-s, x+h, y) - \mathfrak{p}_x(t-s, x, y)) (\mathfrak{p}_x(t-r, x+h, y) - \mathfrak{p}_x(t-r, x, y)) dy. \end{aligned}$$

From (3.5),

$$J(t, s, r, x, y, h) = \frac{2}{\pi} \sum_{k \geq 1} k^2 e^{-k^2(2t-s-r)} (\cos(k(x+h)) - \cos(kx))^2.$$

Using

$$\cos \varphi - \cos \psi = -2 \sin((\varphi - \psi)/2) \sin((\varphi + \psi)/2)$$

and (5.14), and taking $\gamma \in (0, 1/2)$,

$$J(t, s, r, x, y, h) \leq 2h^{2\gamma}(2t-s-r)^{-3/2-\gamma}.$$

Note that

$$3/2 + \gamma < 2.$$

After expanding the square and using the semigroup property, (7.4) becomes

$$\begin{aligned} & \sum_{|\alpha|=n} |v_\alpha(t, x+h) - v_\alpha(t, x)|^2 \\ & \leq 2h^{2\gamma} n! \pi^2 (1 + \sqrt{t})^{2n} \|u_0\|_{L_p((0, \pi))}^2 \\ & \iint_{\mathbb{T}_{0,t}^n \times \mathbb{T}_{0,t}^n} (2t - s_{n-1} - r_{n-1})^{-3/2-\gamma} \prod_{k=1}^{n-2} (s_{k+1} + r_{k+1} - s_k - r_k)^{-1/2} s_1^{-1/2} r_1^{-1/2} ds^n dr^n \quad (7.5) \\ & \leq 2h^{2\gamma} \pi^2 (1 + \sqrt{t})^{2n} \|u_0\|_{L_p((0, \pi))}^2 n! I_n^2(t; 3/4 + (\gamma/2), 1/2) \leq C^n(t, \gamma) n^{-n/2}; \end{aligned}$$

cf. (4.4) and (4.10). Then Proposition 2.1 implies (7.3), completing the proof of Theorem 7.3. □

8. THE FUNDAMENTAL CHAOS SOLUTION

Definition 8.1. *The fundamental chaos solution of (1.4) is the collection of functions*

$$\{\mathfrak{P}_\alpha(t, x, y), \ t > 0, \ x, y \in [0, \pi], \ \alpha \in \mathcal{J}\}$$

defined by

$$\begin{aligned} \mathfrak{P}_{(0)}(t, x, y) &= \mathfrak{p}(t, x, y), \\ \mathfrak{P}_\alpha(t, x, y) &= \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_{(0, \pi)^n} \int_{\mathbb{T}_{0,t}^n} \mathfrak{p}(t - s_n, x, y_n) \mathfrak{m}_{k_{\sigma(n)}}(y_n) \cdots \\ & \cdots \mathfrak{p}(s_2 - s_1, y_2, y_1) \mathfrak{m}_{k_{\sigma(1)}}(y_1) \mathfrak{p}(s_1, y_1, y) ds^n dy^n. \end{aligned} \quad (8.1)$$

The intuition behind this definition is that (8.1) is the chaos solution of (1.4) with initial condition $u_0(x) = \delta(x - y)$. More precisely, it follows from (3.13) that if

$$\mathfrak{P}(t, x, y) = \sum_{\alpha \in \mathcal{J}} \mathfrak{P}_\alpha(t, x, y) \xi_\alpha, \quad (8.2)$$

then

$$u(t, x) = \int_0^\pi \mathfrak{P}(t, x, y) u_0(y) dy \quad (8.3)$$

is the chaos solution of (1.4) with non-random initial condition $u(0, x) = u_0(x)$. Before developing these ideas any further, let us apply the results of Sections 3–7 to the random function \mathfrak{P} .

Theorem 8.2. *The function \mathfrak{P} defined by (8.2) has the following properties:*

$$\mathfrak{P}(t, x, y) \in \bigcap_{q>1} L_{2,q}(W; \mathbb{R}), \quad t > 0, \quad \text{uniformly in } x, y \in [0, \pi]; \quad (8.4)$$

$$\mathfrak{P}(t, x, y) \geq 0, \quad \mathfrak{P}(t, x, y) = \mathfrak{P}(t, y, x), \quad t > 0, \quad x, y \in [0, \pi]; \quad (8.5)$$

$$\mathfrak{P}(\cdot, x, y) \in \mathcal{C}^{3/4-}((\delta, T)), \quad 0 < \delta < T, \quad x, y \in [0, \pi]; \quad (8.6)$$

$$\mathfrak{P}(t, \cdot, y) \in \mathcal{C}^{3/2-}((0, \pi)), \quad t > 0, \quad y \in [0, \pi]. \quad (8.7)$$

Proof. Using (3.7), (4.2), (4.6), and (4.10),

$$\begin{aligned} \sum_{|\alpha|=n} |\mathfrak{P}_\alpha(t, x, y)|^2 &\leq n! \int_{(0, \pi)^n} \int_{\mathbb{T}_{0,t}^n} \int_{\mathbb{T}_{0,t}^n} \left(\mathfrak{p}(t - s_n, x, y_n) \cdots \mathfrak{p}(s_2 - s_1, y_2, y_1) \mathfrak{p}(s_1, y_1, y) \right. \\ &\quad \times \left. \mathfrak{p}(t - r_n, x, y_n) \cdots \mathfrak{p}(r_2 - r_1, y_2, y_1) \mathfrak{p}(r_1, y_1, y) \right) ds^n dr^n dy^n \\ &\leq n! (1 + \sqrt{t})^{2(n+1)} I_n^2(t; 1/4, 1/2) \leq \frac{C^n(t)}{n^{n/2}}, \end{aligned}$$

from which (8.4) follows.

To establish (8.5), note that (8.3) and Theorem 4.5 imply $\mathfrak{P} \geq 0$, whereas, by (3.12), using $\mathfrak{p}(t, x, y) = \mathfrak{p}(t, y, x)$ and a suitable change of the time variables in the integrals,

$$\sum_{|\alpha|=n} |\mathfrak{P}_\alpha(t, x, y) - \mathfrak{P}_\alpha(t, y, x)|^2 = 0, \quad n \geq 1,$$

which implies $\mathfrak{P}(t, x, y) = \mathfrak{P}(t, y, x)$.

To establish (8.6) and (8.7), we compute, for $n \geq 1$,

$$\sum_{|\alpha|=n} |\mathfrak{P}_\alpha(t + h, x, y) - \mathfrak{P}_\alpha(t, x, y)|^2 \leq h^{2\gamma} C^n(t, \gamma) n^{-n/2}, \quad \gamma \in (0, 3/4);$$

cf. (6.7), and

$$\sum_{|\alpha|=n} |\mathfrak{P}_{\alpha,x}(t, x + h, y) - \mathfrak{P}_{\alpha,x}(t, x, y)|^2 \leq h^{2\gamma} C^n(t, \gamma) n^{-n/2}, \quad \gamma \in (0, 1/2);$$

cf. (7.5).

Note that $\mathfrak{P}_{(\mathbf{0})}(t, x, y) = \mathbf{p}(t, x, y)$ is infinitely differentiable in t and x for $t > 0$ but is unbounded as $t \searrow 0$; cf. (3.7).

□

Now we can give full justification of the reason why \mathfrak{P} is natural to call the fundamental chaos solution of equation (1.4).

Theorem 8.3. *If $u_0 \in L_{2,q}(W; L_2((0, \pi)))$ for some $q > 1$, then the chaos solution of (1.4) with initial condition $u(0, x) = u_0(x)$ is*

$$u(t, x) = \int_0^\pi \mathfrak{P}(t, x, y) \diamond u_0(y) dy, \quad (8.8)$$

and

$$u(t, x) \in L_{2,p}(W; \mathbb{R}) \quad (8.9)$$

for every $p < q$, $t > 0$, and $x \in [0, \pi]$.

Proof. Let

$$u_0(x) = \sum_{\alpha \in \mathcal{J}} u_{0,\alpha}(x) \xi_\alpha$$

be the chaos expansion of the initial condition. By definition, the chaos solution of (1.4) is

$$u(t, x) = \sum_{\alpha \in \mathcal{J}} u_\alpha(t, x) \xi_\alpha,$$

where

$$\begin{aligned} \frac{\partial u_{(\mathbf{0})}(t, x)}{\partial t} &= \frac{\partial^2 u_{(\mathbf{0})}(t, x)}{\partial x^2}, \quad u_{(\mathbf{0})}(0, x) = u_{0,(\mathbf{0})}(x); \\ \frac{\partial u_\alpha(t, x)}{\partial t} &= \frac{\partial^2 u_\alpha(t, x)}{\partial x^2} + \sum_k \sqrt{\alpha_k} \mathbf{m}_k(x) u_{\alpha-(k)}(t, x), \\ u_\alpha(0, x) &= u_{0,\alpha}(x), \quad |\alpha| > 0. \end{aligned}$$

By [10, Theorem 9.8], if $u(0, x) = f(x) \xi_\beta$ for some $f \in L_2((0, \pi))$ and $\beta \in \mathcal{J}$, then

$$u(t, x) = \int_0^\pi \mathfrak{P}(t, x, y) \diamond \xi_\beta f(y) dy.$$

Then (8.8) follows by linearity.

Next, given $1 \leq p < q$, take $p' = qp/(q - p)$, so that $p'^{-1} + q^{-1} = p^{-1}$. Then, by (8.4) and [12, Theorem 4.3(a)],

$$\mathfrak{P}(t, x, \cdot) \diamond u_0 \in L_{2,p}(W; L_2((0, \pi))),$$

which implies (8.9).

□

9. FURTHER DIRECTIONS

A natural question is whether the results of Sections 3–8 extend to a more general equation

$$\frac{\partial u(t, x)}{\partial t} = \mathcal{L}u(t, x) + u(t, x) \diamond \dot{W}(x), \quad t > 0, \quad x \in G,$$

where \mathcal{L} is a second-order linear ordinary differential operator and $G \subseteq \mathbb{R}$.

9.1. Equation on a Bounded Interval. Consider a second-order differential operator

$$f \mapsto \mathcal{L}f = \rho(x)f'' + r(x)f' + c(x)f = (\rho f')' + (r - \rho')f' + cf, \quad \rho > 0, \quad x \in (a, b),$$

$$-\infty < a < b < +\infty.$$

A change of variables $f(x) = g(x) \exp\left(-\int \frac{r(x) - \rho'(x)}{2\rho(x)} dx\right)$ leads to the symmetric operator

$$\tilde{\mathcal{L}}g = (\rho g')' + \tilde{c}g,$$

with

$$\tilde{c}(x) = c(x) + \frac{\rho(x)H''(x) + r(x)H'(x)}{H(x)}, \quad H(x) = \exp\left(-\int \frac{r(x) - \rho'(x)}{2\rho(x)} dx\right).$$

The most general form of the (real) homogenous boundary conditions for the operator $\tilde{\mathcal{L}}$ is as follows:

$$AY(a) + BY(b) = 0, \tag{9.1}$$

where $A, B \in \mathbb{R}^{2 \times 2}$, $Y(x) = (g(x) \quad \rho(x)g'(x))^\top$. If the matrix $[A \ B] \in \mathbb{R}^{2 \times 4}$ has rank 2 and

$$AEA^\top = BEB^\top, \quad E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{9.2}$$

then boundary conditions (9.1) allow a self-adjoint extension of the operator $\tilde{\mathcal{L}}$ to $L_2((a, b))$; cf. [19, Section 4.2]. Particular cases of (9.1) satisfying (9.2) are **separated boundary conditions**

$$c_1g(a) + c_2\rho(a)g'(a) = 0, \quad c_3g(b) + c_4\rho(b)g'(b) = 0,$$

when both matrix products in (9.2) are zero, and **periodic boundary conditions**

$$g(a) = g(b), \quad g'(a) = g'(b),$$

when $A = B$.

Consider the eigenvalue problem for $\tilde{\mathcal{L}}$:

$$-\tilde{\mathcal{L}}\mathbf{m}_k(x) = \lambda_k^2 \mathbf{m}_k, \quad k = 1, 2, \dots$$

The following properties of the eigenvalues λ_k and the eigenfunctions \mathbf{m}_k ensure that the results of Sections 3–8 extend to equation

$$u_t = \tilde{\mathcal{L}}u + u \diamond \dot{W}(x).$$

[EVS] The eigenvalues λ_k satisfy

$$\lim_{k \rightarrow \infty} \frac{\lambda_k}{k^2} = \bar{\lambda} > 0;$$

[CEF] The set of normalized eigenfunctions $\{\mathbf{m}_k, k \geq 1\}$ is complete in $L_2((a, b))$;

[MP] The kernel

$$\mathbf{p}(t, x, y) = \sum_{k=1}^{\infty} e^{-\lambda_k^2 t} \mathbf{m}_k(x) \mathbf{m}_k(y)$$

of the semi-group generated by $\tilde{\mathcal{L}}$ is non-negative: $\mathbf{p}(t, x, y) \geq 0, t > 0, x, y \in (a, b)$;

[UB] The eigenfunctions \mathbf{m}_k are uniformly bounded:

$$\sup_{k \geq 1, x \in (a, b)} |\mathbf{m}_k(x)| < \infty.$$

The corresponding computations, although not necessarily trivial, are essentially equivalent to what was done in Sections 3–8. In the case of additive noise, the results of [2] help with identification of the diffusion process that, similar to the Brownian bridge in Remark 5.3, compensates the spatial derivative of the solution to a smooth function.

There are general sufficient conditions for [EVS] and [CEF] to hold; cf. [19, Theorems 4.3.1 and 4.6.2]. The maximum principle [MP] means special restrictions on the boundary conditions; these restrictions are, in general, not related to (9.1); cf. [2, Theorem 12]. Condition [UB] appears to be the most difficult to verify without additional information about the operator $\tilde{\mathcal{L}}$.

9.2. Equation on the Line. All the results of Sections 3–8 extend to the chaos solution of the heat equation

$$u_t = u_{xx} + u \diamond \dot{W}(x), \quad t > 0, \quad x \in \mathbb{R},$$

with suitable initial condition $u(0, x) = u_0(x)$. In the case of additive noise, a two-sided Brownian motion compensates the space derivative of the solution to a smooth function.

Further extensions, to the equation

$$u_t = \mathcal{L}u + u \diamond \dot{W}(x),$$

are also possible but might require additional effort.

For example, let $\mathcal{L}f = (c(x)f'(x))'$ with a measurable function $c(x)$ such that

$$c_1 \leq c(x) \leq c_2, \quad \text{for all } x \in \mathbb{R},$$

for some constants $c_1, c_2 > 0$. Then [16, Theorems 4.1.11 and 4.2.9] the kernel of the the corresponding semi-group satisfies

$$\frac{c_1}{\sqrt{t}} \exp\left(-\frac{(x-y)^2}{a_1 t}\right) \leq \mathbf{p}(t, x, y) = \mathbf{p}(t, y, x) \leq \frac{c_2}{\sqrt{t}} \exp\left(-\frac{(x-y)^2}{a_2 t}\right)$$

for some positive numbers $\mathbf{c}_1, \mathbf{c}_2, \mathbf{a}_1, \mathbf{a}_2$, which is enough to carry out the computations from Sections 3 and 4. Additional regularity of the chaos solution requires more delicate bounds on $\mathbf{p}_x(t, x, y)$ and $\mathbf{p}_t(t, x, y)$.

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